

ON THE IRREDUCIBLE REPRESENTATIONS OF SOLUBLE GROUPS OF FINITE RANK

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ABSTRACT. We obtained some sufficient and necessary conditions of existence of faithful irreducible representations of a soluble group G of finite rank over a field k . It was shown that the existence of such representations strongly depends on construction of the socle of the group G . The situation is especially complicated in the case where the field k is locally finite.

1. INTRODUCTION

We recall that a group G has finite (Prufer) rank if there is an integer r such that each finitely generated subgroup of G can be generated by r elements; its rank $r(G)$ is then the least integer r with this property. A group G is said to be polycyclic if it has a finite series in which each factor is cyclic.

Let G be a group, the subgroup $Soc(G)$ of G generated by all its minima normal subgroups is said to be the socle of the group G (if the group G has no minimal normal subgroups then $Soc(G) = 1$). The subgroup $abSoc(G)$ of the group G generated by all its minima normal abelian subgroups is said to be the abelian socle of the group G (if the group G has no minimal abelian normal subgroups then $abSoc(G) = 1$). If the group G is soluble then $abSoc(G) = Soc(G)$.

A nontrivial normal subgroup E of a group G is said to be essential if $E \cap N \neq 1$ for any nontrivial normal subgroup N of G .

If B is an abelian group of finite rank then the spectrum $Sp(B)$ of the group B is the set of prime numbers p such that the group B has an infinite p -section.

It was proved in [7] that if a polycyclic group G has a faithful irreducible representation over a locally finite field k then the group G is finite. However, infinite locally polycyclic groups of finite rank may have faithful irreducible representations over a locally finite field k . Moreover, in [8] we found necessary and sufficient conditions of existence of faithful irreducible representations of locally polycyclic groups of finite rank over a locally finite field k .

In [4] Gaschutz discovered a critical role of $Soc(G)$ in the theory of representations of finite groups. We denote by $\mathfrak{S}_{\mathfrak{f}}$ a class of all groups G such that $Soc(G)$ is an essential normal subgroup of G and all minimal normal subgroups of G are finite. We should note that the class $\mathfrak{S}_{\mathfrak{f}}$ is rather large and contains locally normal groups and torsion groups of finite rank. In theorems 1 and 2 of [9] we proved that a group $G \in \mathfrak{S}_{\mathfrak{f}}$ has an irreducible faithful representation over a field k if and only if $chark \notin \pi(abSoc(G))$ and one of the following equivalent conditions holds:

- (i) $abSoc(G)$ has a subgroup H such that $abSoc(G)/H$ is a locally cyclic group and H contains no nontrivial G -invariant subgroups;
- (ii) $abSoc(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where G acts on $abSoc(G)$ by conjugations.

We should note that in the case of torsion soluble groups of finite rank such criteria were obtained in theorems 2 and 3 of [8]. Thus, the structure of $abSoc(G)$ is the most important for existence of faithful irreducible representations of locally normal and, in particular, finite groups.

In the presented paper we are searching necessary and sufficient conditions of existence of faithful irreducible representations of soluble groups of finite rank over a field k . In theorem 5.1 is proved that if a soluble group of finite rank G has a faithful irreducible representation over a field k then $Soc(G)$ of the group G is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $Soc(G)$ by conjugations, and $chark \notin \pi(Soc(G))$. Theorem 5.2 shows that if the field k is not locally finite then the above condition on $Soc(G)$ is also sufficient for existence of an irreducible faithful representation of G over k . By theorem 5.3, in the case where the field k is locally finite the condition on $Soc(G)$ is also a criterion of existence of an irreducible faithful representation of G over k with an additional assumption that $Sp(B) \not\subset \{chark\}$ for any nontrivial torsion-free abelian normal subgroup B of G .

However, the condition $Sp(B) \not\subset \{chark\}$ for any nontrivial abelian torsion-free normal subgroup of G is not necessary for existence of faithful irreducible representations of soluble groups of finite rank over a locally finite field. It follows from a result by Wherfritz [10] in which a simple kG -module W was constructed such that $C_G(W) = 1$, where k is a field of order p , $G = B\lambda \langle g \rangle$ is a torsion-free soluble group of rank 2 and $Sp(B) = \{p\}$. So, in the case of locally finite fields the situation is much more complicated.

2. ON DIRECT SUMS OF SOME JUST INFINITE MODULES

An abelian group is said to be minimax if it has a finite series each of whose factor is either cyclic or quasi-cyclic. It is easy to note that for any abelian minimax group A the set $Sp(A)$ is finite.

Let R be a ring, an R -module A is said to be just infinite (or R -ji-module for shortness) if A is infinite and for any proper submodule V of A the quotient module A/V is finite. We also omit R in the notation if the structure of the ring R is not important.

We should emphasize that below we always assume that any R -ji-module is torsion-free minimax as an abelian group.

Lemma 2.1. *Let $A = \bigoplus_{i=1}^n A_i$, where A_i are ji-modules. Then:*

- (i) *any nonzero submodule of A contains a nonzero ji-submodule;*
- (ii) *the set of all ji-submodules of A is countable.*

Proof. (i) Let X be a nonzero submodule of A . Evidently, the module A has a finite series each of whose quotient is a ji-module. The intersection of this series with X give us a finite series of submodules of X each of whose quotient is either zero or a ji-module and the assertion follows.

(ii) The proof is by induction on n . Let $n = 1$ then A is a ji-module and for any proper submodule B of A the quotient module A/B is finite. Therefore there is an integer n such that $An \leq B$ and, as the set of integers n is countable and for each such n there is only finite set of submodules B of A such that $An \leq B$, the set of all submodules of A is an union of countable many of finite subsets. So, the set of all ji-submodules of A is countable.

Consider now the general case then $A = C \oplus D$, where $C = A_1$ and $D = \bigoplus_{i=2}^n A_i$. By the induction hypothesis, the set \mathfrak{R} of all ji-submodules of C and D is countable. Let B be a ji-submodule of A , if $C \cap B \neq 0$ then the quotient group $B/(C \cap B)$ is torsion-free because so is the quotient group A/C and, as B is a ji-submodule, it implies that $B \leq C$. The same arguments show that if $D \cap B \neq 0$ then $B \leq D$. So, if $C \cap B \neq 0$ or $D \cap B \neq 0$ then $B \in \mathfrak{R}$.

Let \mathfrak{S} be the set of all ji-submodules of A such that $C \cap B = 0$ and $D \cap B = 0$. Since the set \mathfrak{R} is countable, it is sufficient to show that so is \mathfrak{S} . Thus, we can assume that $B \in \mathfrak{S}$. Then the mappings $\varphi : B \rightarrow \text{Pr}_C B$ and $\phi : B \rightarrow \text{Pr}_D B$ given by $\varphi : b \mapsto \text{Pr}_C b$ and $\phi : b \mapsto \text{Pr}_D b$ are module isomorphisms. Since the set \mathfrak{R} is countable, the set of submodules $U \oplus V$ of A , where $U \leq C$ and $V \leq D$ are isomorphic ji-submodules, is countable. Then it is easy to note that it is sufficient to show countability of the set \mathfrak{N} of all ji-submodules B of $(W)_1 \oplus (W)_2$

such that $(W)_1 \cap B = (W)_2 \cap B = 0$ and $W = \text{Pr}_{(W)_1} B = \text{Pr}_{(W)_2} B$, where W is a ji -module. Let $B \in \aleph$, as $(W)_1 \cap B = (W)_2 \cap B = 0$ and $W = \text{Pr}_{(W)_1} B = \text{Pr}_{(W)_2} B$, for any $w \in (W)_1 = W$ there is a unique $v \in (W)_2 = W$ such that $w+v \in B$. It is easy to note that the mapping $\psi_B : W \rightarrow W$ given by $\psi_B : w \mapsto v$ is a group automorphism of W . So, we have a mapping $\eta : \aleph \rightarrow \text{Aut}W$ given by $B \mapsto \psi_B$. If $\psi_B = \psi_K$ for some $K, B \in \aleph$ then for any $w \in (W)_1 = W$ and $v \in (W)_2 = W$ we have $w+v \in B$ if and only if $w+v \in K$ and it easily implies that $K = B$. So, η is an injection. Since W is a torsion-free abelian group of finite rank n , it is well known that $\text{Aut}W \leq GL_n(\mathbb{Q})$. Then, as the set $GL_n(\mathbb{Q})$ is countable, so is $\text{Aut}W$ and, as we have an injection $\eta : \aleph \rightarrow \text{Aut}W$, the set \aleph is countable. \square

Lemma 2.2. *Let A_i be quasicyclic p -groups, where $i = 1, 2$, and $A = \bigoplus_{i=1}^2 A_i$. Then the cardinality of the set of all subgroups B of A which defines the quasicyclic quotient group A/B is continuum.*

Proof. As for any quasicyclic subgroup B of A the quotient group A/B is quasicyclic, it is sufficient to show that the cardinality of the set of all quasicyclic subgroups of A is continuum. Let \aleph be the set of all quasicyclic p -subgroups B of A such that $B \cap A_1 = B \cap A_2 = 0$. Then it is sufficient to show that the cardinality of the set \aleph is continuum. Let $B \in \aleph$ then any element of B may be uniquely written in the form $a = a_1 + a_2$, where $a_1 = \text{Pr}_{A_1}(a)$ and $a_2 = \text{Pr}_{A_2}(a)$. If $\alpha \in \text{Aut}(A_2)$ then elements $a_1 + \alpha(a_2)$ form another subgroup $B_\alpha \in \aleph$. Let $\alpha, \beta \in \text{Aut}(A_2)$ if $B_\alpha = B_\beta$ then $\alpha(a_2) = \beta(a_2)$ for all $a \in B$ and it implies that $\alpha = \beta$. Thus, we have an injection $\alpha \mapsto B_\alpha$ of $\text{Aut}(A_2)$ into \aleph . It is well known that $\text{Aut}(A_2)$ is isomorphic to the group $U(J_p)$ of units of the ring J_p of p -adic integers. As the cardinality of $U(J_p)$ is continuum, so is the cardinality of \aleph . \square

Proposition 2.1. *Let $A = \bigoplus_{i=1}^n A_i$, where A_i are ji -modules and let p be a prime number such that $p \in \text{Sp}(A_i)$ for all $i = \overline{1, n}$. Then the group A has a subgroup B such that A/B is a quasicyclic p -group and B does not contain nonzero submodules.*

Proof. The proof is by induction on n . Let $n = 1$ then A is a ji -module and, as $p \in \text{Sp}(A_i)$, the group A has a subgroup B such that A/B is a quasicyclic p -subgroup. It follows from the definition of ji -modules that B does not contain nonzero submodules.

Consider now the general case then $A = C \oplus D$, where $C = A_1$ and $D = \bigoplus_{i=2}^n A_i$. By the induction hypothesis, C contains a subgroup C_1 and D contains a subgroup D_1 which do not contain nonzero submodules and such that C/C_1 and D/D_1 are quasicyclic p -groups. Put

$B_1 = C_1 \oplus D_1$. Show that the subgroup B_1 does not contain nonzero submodules. Let X be a submodule of B_1 . Then $\text{Pr}_{C_1}(X)$ is a submodule of C_1 and hence, as C_1 does not contain nonzero submodules, we see that $\text{Pr}_{C_1}(X) = 0$. The same arguments show that $\text{Pr}_{D_1}(X) = 0$ and hence $X = 0$. Thus, B_1 does not contain nonzero submodules.

Evidently, $A/B_1 = P_1 \oplus P_2$, where $P_1 = C/C_1$ and $P_2 = D/D_1$ are quasicyclic groups. Let \mathfrak{R} be the set of all subgroups E of A such that $B_1 \leq E$ and A/E is a quasicyclic p -group. By lemma 2.2, the cardinality of the set \mathfrak{R} is continuum. Let \mathfrak{N} be the set of all ji -modules of A and let $\mathfrak{S} = \{(K + B_1)/B_1 \mid K \in \mathfrak{N}\}$ then it follows from lemma 2.1(ii) that the set \mathfrak{S} is countable. If for some $K \in \mathfrak{N}$ the quotient group $(K + B_1)/B_1$ is finite then $|K_1/(K \cap B_1)| = n$ for some integer n and hence $Kn \leq B_1$. But it is impossible because B_1 does not contain nonzero submodules. Thus, the quotient group $(K + B_1)/B_1$ is infinite for any subgroup $K \in \mathfrak{N}$ and hence for any subgroup $K \in \mathfrak{N}$ either $A = K + B_1$ or $A/(K + B_1)$ is a quasicyclic p -group. Since the set of all subgroups of a quasicyclic p -group is countable, it implies that for any $K \in \mathfrak{N}$ the set \mathfrak{R}_K of all subgroups $U \in \mathfrak{R}$ such that $K + B_1 \leq U$ is countable. Therefore, as the set \mathfrak{N} is countable, the set of all subgroups $U \in \mathfrak{R}$ which for some $K \in \mathfrak{N}$ contain the subgroup $K + B_1$ is countable. As the cardinality of the set \mathfrak{R} is continuum, it implies that there is a subgroup $B \in \mathfrak{R}$ such that A/B is a quasicyclic p -subgroup and B does not contain subgroups from the set \mathfrak{N} . By lemma 2.1(i) any nonzero submodule of A contains a submodule from \mathfrak{N} and hence the subgroup B does not contain nonzero submodules. \square

A subgroup H of an abelian group A is said to be dense if the quotient group A/H is torsion.

Proposition 2.2. *Let $A = \bigoplus_{i=1}^n A_i$, where A_i are R - ji -modules, and let p be a prime number. Suppose that $Sp(A_i) \not\subset \{p\}$ for all $i = \overline{1, n}$ then the group A has a dense subgroup H such that A/H is locally cyclic p' -group and H does not contain nonzero submodules.*

Proof. As $Sp(A_i) \not\subset \{p\}$ for all $i = \overline{1, n}$, there is a prime number $p_1 \neq p$ such that $p_1 \in Sp(A_i)$ for some i . Let B_1 be the direct sum of all A_i such that $p_1 \in Sp(A_i)$. If $B_1 \neq A$ then, as $Sp(A_i) \not\subset \{p\}$ for all $i = \overline{1, n}$, there is a prime number $p_2 \neq p_1 \neq p$ such that $p_2 \in Sp(A_i)$ for some i and $A_i \not\subset B_1$. Let B_2 be the direct sum of all A_i such that $p_2 \in Sp(A_i)$ and $A_i \not\subset B_1$. Continuing this process we obtain a direct decomposition $A = \bigoplus_{i=1}^m B_i$ of the module A , where B_i is a direct sum of ji -modules each of whose spectrum contains prime number $p_i \neq p$ and $p_i \notin Sp(B_j)$ if $i < j$. By proposition 2.1, each submodule B_i contains a subgroup

H_i such that B_i/H_i is a quasicyclic p_i -group and H_i does not contain nonzero submodules.

Put $H = \bigoplus_{i=1}^m H_i$. Evidently, H is a dense subgroup of A . As all the prime numbers $p_i \neq p$ are different, it is easy to note that A/H is locally cyclic p' -group. Show that H does not contain nonzero submodules. Suppose that H contains a nonzero submodule D . Any nonzero element $d \in D$ may be unequally written in the form $d = \sum_{i=1}^m d_i$, where $d_i \in H_i$. Then we can chose a nonzero element $d \in D$ with the minimal number k of nonzero summands $d_i \in H_i$ in the representation $d = \sum_{i=1}^m d_i$ of the element d . So, $d = \sum_{j=1}^k d_{i_j}$, where $0 \neq d_{i_j} \in H_{i_j}$, and it follows from the minimality of k that for any $\alpha \in R$ we have $d\alpha = 0$ if and only if $d_{i_j}\alpha = 0$ for each d_{i_j} . It easily implies that the mapping $\varphi_{i_j} : dR \rightarrow d_{i_j}R$ given by $\varphi_{i_j} : d\alpha \mapsto d_{i_j}\alpha$ is a R -module isomorphism for each d_{i_j} . Therefore, all modules $d_{i_j}R$ are isomorphic but this is impossible, because by the construction of submodules B_i , $p_{i_1} \in Sp(d_{i_1}R)$ and $p_{i_1} \notin Sp(d_{i_j}R)$ for any $i_j > i_1$. \square

3. ESSENTIAL NORMAL SUBGROUPS OF SOLUBLE GROUPS OF FINITE RANK

Let G be an infinite group, we say that an infinite normal subgroup A of the group G is G -just-infinite (or G -ji-subgroup for shortness) if $|A : B| < \infty$ for any proper G -invariant subgroup B of A .

Lemma 3.1. *If a soluble group G of finite rank has a nontrivial torsion-free normal subgroup N then N has a torsion-free minimax abelian G -just-infinite subgroup.*

Proof. If the group G has a nontrivial torsion-free normal subgroup then it has a nontrivial abelian torsion-free normal subgroup A . We can consider A as a $\mathbb{Z}G$ -module, where G acts on A by conjugations, and there is no harm in assuming that $C_G(A) = 1$. Let $B = A \otimes_{\mathbb{Z}} \mathbb{Q}$ be the divisible hull of A . It follows from the results of [2] that the group G has a finite series of normal subgroups $1 \leq N \leq H \leq G$, where the quotient group H/N is finitely generated abelian, the quotient group G/H is finite, and the group B has a nontrivial G -invariant subgroup C centralized by N . Then replacing A by $A \cap C$ we can assume that $G/C_G(A)$ is finitely generated abelian-by-finite. It follows from lemma 5.1 of [5] that $a\mathbb{Z}G$ is a minimax group for any $0 \neq a \in A$ and replacing A by $a\mathbb{Z}G$ we can assume that A is a minimax group. Since, by [11], for any descending chain $\{A_i\}$ of subgroups of A there is an integer n such that $|A_i : A_{i+1}| < \infty$ for all $i \geq n$, it is easy to show that there is a nontrivial G -just-infinite subgroup $D \leq A$. \square

Lemma 3.2. *Let G be a soluble group of finite rank which has a non-trivial torsion-free normal subgroup. Then the group G has a torsion-free minimax abelian normal subgroup $jiSoc(G) \neq 1$ such that $jiSoc(G)$ is a direct product of finitely many of torsion-free abelian minimax G -just-infinite subgroups and $jiSoc(G) \cap N \neq 1$ for any nontrivial torsion-free normal subgroup N of the group G .*

Proof. We construct the subgroup $jiSoc(G) = S$ by induction. By lemma 2.1, the group G has a torsion-free minimax abelian G -just-infinite subgroup A and we put $S_1 = A$. If there is a nontrivial torsion-free normal subgroup N of the group G such that $S_1 \cap N = 1$ then it follows from lemma 3.1 that there is a torsion-free minimax abelian G -just-infinite subgroup $1 \neq A_1 \leq N$ such that $S_1 \cap A_1 = 1$ and we put $S_2 = S_1 \times A_1$. Continuing this process we should note that it is terminated because the group G has finite rank and hence the subgroup $jiSoc(G)$ does exist. \square

If the group G has no nontrivial torsion-free normal subgroup then we put $jiSoc(G) = 1$, it follows from the above lemma that $jiSoc(G) \neq 1$ if and only if the group G has a nontrivial torsion-free normal subgroup. Certainly, the subgroup $jiSoc(G)$ is not defined uniquely.

Proposition 3.1. *Let G be a soluble group of finite rank and let p be a prime number. Suppose that $Sp(B) \not\subseteq \{p\}$ for any nontrivial abelian torsion-free normal subgroup B of G . If $jiSoc(G) \neq 1$ then $jiSoc(G)$ has a dense subgroup H such that $jiSoc(G)/H$ is a locally cyclic torsion p' -group and H does not contains nontrivial G -invariant subgroups.*

Proof. As $Sp(B) \not\subseteq \{p\}$ for any nontrivial abelian torsion-free normal subgroup B of G , it follows from lemma 2.1 that $Sp(B) \not\subseteq \{p\}$ for any abelian torsion-free G -just-infinite subgroup B of G . We can consider $jiSoc(G)$ as a $\mathbb{Z}G$ -module, where the group G acts on $jiSoc(G)$ by conjugations. Then $jiSoc(G) = \bigoplus_{i=1}^n A_i$, where A_i are $\mathbb{Z}G$ -j.i.-modules. Therefore, $jiSoc(G)$ and p meet the conditions of proposition 2.2 and the assertion follows. \square

Proposition 3.2. *Let G be a soluble group of finite. Then $jiSoc(G) \times Soc(G)$ is an essential subgroup of G and for any normal subgroup $1 \neq N \leq G$ either $jiSoc(G) \cap N \neq 1$ or $Soc(G) \cap N \neq 1$.*

Proof. The subgroup $N \leq G$ contains a nontrivial abelian G -invariant subgroup and hence we can assume that the subgroup N is abelian. If the torsion subgroup T of N is trivial than it follows from lemma 3.2 that $jiSoc(G) \cap N \neq 1$. If $T \neq 1$ then, as $r(G) < \infty$, T contains a nontrivial finite G -invariant subgroup and hence $Soc(G) \cap N \neq 1$. \square

4. ESSENTIAL SUBGROUPS AND REPRESENTATIONS OF GROUPS

Proposition 4.1. *Let k be a field and $H = \times_{i=1}^n H_i$ be a group. Let φ_i be an irreducible representation of the group H_i over the field k , where $i = \overline{1, n}$. Then there is an irreducible representation φ of the group H over the field k such that $\text{Ker}\varphi \cap H_i = \text{Ker}\varphi_i$, where $i = \overline{1, n}$.*

Proof. Let M_i be a kH_i -module of the representation φ_i then, as the module M_i is irreducible, there is a generator a_i of the module M_i , where $i = \overline{1, n}$. We can consider M_i as a kH -module, where H_j acts on M_i trivially for any $j \neq i$.

Put $M = M_1 \otimes_F M_2 \otimes_F \dots \otimes_F M_n$, where H acts on M as the following : $(m_1 \otimes m_2 \otimes \dots \otimes m_n)g = m_1 g \otimes m_2 g \otimes \dots \otimes m_n g$ (see [6], Chap. XVIII, §2). Then it is not difficult to show that the kH -module M is generated by $a = a_1 \otimes a_2 \otimes \dots \otimes a_n$ and $\text{Ann}_{FH_i}(a) = \text{Ann}_{FH_i}(a_i)$ for any $i = \overline{1, n}$. So, $akH_i \simeq M_i$ and we can assume that $M_i = akH_i$, where $i = \overline{1, n}$.

Let L be a maximal submodule of M . Put $W = M/L$ and let φ be an irreducible representation of the group H induced by action of H on the kH -module W . Since the kH_i -module $M_i = akH_i$ is irreducible, $M_i \cap L = 0$ and hence we can assume that $M_i \leq W$. Therefore, $\text{Ker}\varphi_i = C_{H_i}(M_i) \geq C_{H_i}(W) = \text{Ker}\varphi \cap H_i$. On the other hand, it follows from the definition of action of H_i on M that $\text{Ker}\varphi_i = C_{H_i}(M_i) \leq C_{H_i}(M) \leq C_{H_i}(W) = \text{Ker}\varphi \cap H_i$ and we can conclude that $(\text{Ker}\varphi \cap H_i) = \text{Ker}\varphi_i$, where $i = \overline{1, n}$. \square

Proposition 4.2. *Let G be a group with a normal subgroup $H = \times_{i=1}^n H_i$, where all H_i are normal subgroups of G , such that $N \cap H_i \neq 1$ for any normal subgroup $1 \neq N \leq G$ and for some subgroup H_i . Let k be a field and suppose that each subgroup H_i has an irreducible representation φ_i over the field k such that $\text{Ker}\varphi_i$ does not contain nontrivial G -invariant subgroups. Then the group G has an irreducible faithful representation over the field k .*

Proof. By proposition 4.1, there is an irreducible representation φ of the subgroup H over the field k such that $\text{Ker}\varphi \cap H_i = \text{Ker}\varphi_i$, where $i = \overline{1, n}$. Let M be a module of the representation φ then $M \cong kH/J$, where J is a maximal ideal of kH . Let I be a maximal ideal of kG such that $J \leq I$ then $I \cap kH = J$ and $W = kG/I$ is an irreducible kG -module. Let ϕ be a representation of the group G over the field k induced by action of G on W . As $I \cap kH = J$, we can assume that $M \leq W$ and hence $\text{Ker}\phi \cap H \leq \text{Ker}\varphi$. Then, as $\text{Ker}\varphi \cap H_i = \text{Ker}\varphi_i$, we can conclude that $\text{Ker}\phi \cap H_i \leq \text{Ker}\varphi_i$ for any $i = \overline{1, n}$. If $\text{Ker}\phi \neq 1$ then $\text{Ker}\phi \cap H_i \neq 1$ for some i but it is impossible because

$\text{Ker}\phi \cap H_i \leq \text{Ker}\varphi_i$ and $\text{Ker}\varphi_i$ does not contain nontrivial G -invariant subgroups. \square

Lemma 4.1. *Let G be a group and let k be a field. Let M be a simple kG -module such that $C_G(M) = 1$. Then :*

- (i) *M contains a simple kH -module for any subgroup $H \leq G$ of finite index in G ;*
- (ii) *$C_M(N) = 0$ for any nontrivial normal subgroup N of G ;*
- (iii) *if A is a central subgroup of G then $\text{Ann}_{kA}(M)$ is a prime ideal of kA ;*
- (iv) *if A is an elementary abelian normal p -subgroup of G such that $|G : C_G(A)| < \infty$ then $p \neq \text{char} k$ and A contains a subgroup H such that A/H is a cyclic group and H contains no nontrivial G -invariant subgroups.*

Proof. (i) Let $K = \bigcap_{g \in G} H^g$ then K is a normal subgroup of finite index in G such that $K \leq H$. Let V be a maximal kK -submodule of M . As $|G : K| < \infty$, it is easy to note that the set $\{Vg | g \in G\}$ of kK -submodules of M is finite. Since $\bigcap_{g \in G} Vg \neq M$ is a kG -submodule of M , we see that $\bigcap_{g \in G} Vg = 0$. Then, by Remak theorem, $M \leq \bigoplus_{g \in G} M/Vg$. As the set $\{Vg | g \in G\}$ is finite, so is the set $\{M/Vg | g \in G\}$ of simple kK -modules M/Vg and hence, as $M \leq \bigoplus_{g \in G} M/Vg$, we can conclude that M is an artinian kK -module. Therefore, as $K \leq H$, M is an artinian kH -module and hence M contains a simple kH -module.

(ii) Suppose that $C_M(N) = W \neq 0$. Since $wgh = w(ghg^{-1})g = wg$ for any $w \in W$, $h \in N$ and $g \in G$, we can conclude that W is a nonzero kG -submodule of the simple module M . Therefore, $C_M(N) = W = M$ and hence $1 \neq N \leq C_G(M)$ but this is impossible because $C_G(M) = 1$.

(iii) As A is a central subgroup of G , it is not difficult to show that Ma is a submodule of M for any element $a \in kA$. Then, as the module M is simple, it is easy to note that either $Ma = 0$ or $Ma = M$ for any element $a \in kA$. It implies that if $Mab = 0$ for some $a, b \in kA$ then either $a \in \text{Ann}_{kA}(M)$ or $b \in \text{Ann}_{kA}(M)$. Thus $\text{Ann}_{kA}(M)$ is a prime ideal of kA .

(iv) Put $C = C_G(A)$ then it follows from (i) that M contains a simple kC -module W . It follows from (iii) that $\text{Ann}_{kA}(W)$ is a prime ideal of kA . If $p = \text{char} k$ then kA has a unique prime fundamental ideal $I = \langle 1 - g | g \in A \rangle$. But then $0 \neq W \leq C_M(A)$ that contradicts (ii). Thus, $p \neq \text{char} k$.

Since $\text{Ann}_{kA}(W)$ is a prime ideal of kA , the quotient group $A/C_A(W)$ is a subgroup of the multiplicative group of the field of fractions of the domain $kA/\text{Ann}_{kA}(W)$. Then, as A is an elementary abelian p -group, it follows from theorem 127.3 of [3] that the quotient group $A/C_A(W)$

is cyclic. If $C_A(W)$ contains a G -invariant nontrivial subgroup K then $0 \neq W \leq C_M(K)$ that contradicts (ii). Thus, we can put $H = C_A(W)$. \square

Proposition 4.3. *Let G be a group such that all minimal abelian normal subgroups of G are finite and let k be a field. If the group G has a faithful irreducible representation φ over the field k then $\text{char } k \notin \pi(\text{abSoc}(G))$ and one of the following equivalent conditions holds:*

- (i) *the abelian socle $\text{abSoc}(G)$ contains a subgroup H such that the quotient group $\text{abSoc}(G)/H$ is locally cyclic and H contains no nontrivial G -invariant subgroups;*
- (ii) *the abelian socle $\text{abSoc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $\text{abSoc}(G)$ by conjugations.*

Proof. Let M be a simple kG -module of the representation φ then $C_G(M) = 1$. Put $\text{char } k = q$ if $\text{char } k \in \pi(\text{abSoc}(G))$ then the group G has a finite nontrivial normal abelian q -subgroup A . As $|A| < \infty$, we can conclude that $|G : C_G(A)| < \infty$ but it contradicts lemma 4.1(iv). Thus, $\text{char } k \notin \pi(\text{abSoc}(G))$.

(i) We can consider $A = \text{abSoc}(G)$ as a semi-simple $\mathbb{Z}G$ -module all of whose simple submodules are finite, where the group G acts on $\text{abSoc}(G)$ by conjugations. Let $B = \bigoplus_{i \in I} B_i$ be an isotype component of A (see [1], Chap. VIII, 3), where B_i are isomorphic simple $\mathbb{Z}G$ -modules, then B is an elementary abelian p -group. Since the modules B_i are isomorphic and $B = \bigoplus_{i \in I} B_i$, it is easy to show that $C_G(B_i) = C_G(B)$ for all $i \in I$ and, as all submodules B_i are finite, we can conclude that $|G : C_G(B)| < \infty$.

Let $A_p = \bigoplus_{i \in I} A_i$ be a Sylow p -component of A , where A_i are isotype components of A_p . As it was shown above, $|G : C_G(A_i)| < \infty$ and hence, by lemma 4.1(iv), each component A_i has a subgroup H_i which contains no nontrivial G -invariant subgroups and such that $|A_i/H_i| = p$. Then, as A_i is an elementary abelian p -group, there is an element $a_i \in A_i$, such that $A_i = H_i \oplus \langle a_i \rangle$. We fix an index $i_1 \in I$ and put $H_p = \langle \{a_i - a_{i_1} \mid i_1 \neq i\} \cup \{H_i \mid i \in I\} \rangle$. Then it is not difficult to note that $|A_p/H_p| = p$ and $H_p \cap A_i = H_i$.

Suppose that H_p contains a nontrivial $\mathbb{Z}G$ -submodule then H_p contains a simple submodule T . By the definition of isotype components, $T \leq A_i$ for some $i \in I$ and it implies that $T \leq H_p \cap A_i$. However, $H_p \cap A_i = H_i$ and hence $T \leq H_i$ but this is impossible because H_i contains no nonzero $\mathbb{Z}G$ -submodules. Thus, each Sylow p -component A_p of A has a subgroup H_p which contains no nonzero $\mathbb{Z}G$ -submodules

and such that the quotient group A_p/H_p is cyclic. Put $H = \bigoplus_{p \in \pi(A)} H_p$ then it is not difficult to show that the quotient group A/H is locally cyclic and H contains no nontrivial G -invariant subgroups.

(ii) It follows from lemma 7 of [8] that conditions (i) and (ii) are equivalent. \square

5. SOME NECESSARY AND SUFFICIENT CONDITIONS OF EXISTENCE OF FAITHFUL IRREDUCIBLE REPRESENTATIONS OF SOLUBLE GROUPS OF FINITE RANK

Lemma 5.1. *Let A be a torsion-free abelian group of finite rank and let k be a field.*

(i) *if the field k is not locally finite then the group A has a faithful irreducible representations over the field k ;*

(ii) *if the field k is locally finite of characteristic p and the group A has a dense subgroup H such that A/H is a locally cyclic p' -group then the group A has an irreducible representations φ over the field k such that $\text{Ker}\varphi = H$.*

Proof. Let k^* be the multiplicative group of the field k , let \hat{k} be the algebraic closure of the field k and let \hat{k}^* be the multiplicative group of the field \hat{k} . At first, we note that any group homomorphism $\varphi : A \rightarrow \hat{k}^*$ may be continued to a ring homomorphism $\varphi : kA \rightarrow \hat{k}$ given by $\varphi : \sum_{a \in A} k_a a \mapsto \sum_{a \in A} k_a \varphi(a)$, where $k_a \in k$. Besides, $\varphi(kA)$ is a subfield of \hat{k} because $k \leq \varphi(kA)$ and for any element $a \in \hat{k}$ the subring of \hat{k} generated by k and a is a subfield of \hat{k} (see [6], Chap. VI, Proposition 3). Therefore, $\varphi(kA)$ is an irreducible kA -module, where action of $\alpha \in kA$ on $\varphi(kA)$ is defined by multiplication of elements of $\varphi(kA)$ by $\varphi(\alpha)$, and φ is an irreducible representation of the group A over the field k .

(i) Since the field k is not locally finite, either $\mathbb{Q} \leq k$ or $f(t) \leq k$, where $f(t)$ is the field of fractions of the group algebra of infinite cyclic group $\langle t \rangle$ over a finite field f . It is well known that the multiplicative groups \mathbb{Q}^* and $f(t)^*$ are not torsion. Then it follows from theorem 127.3 of [3] that \hat{k}^* has a torsion-free divisible subgroup of infinite rank. Therefore, it is not difficult to note that there is a group monomorphism $\varphi : A \rightarrow \hat{k}^*$ and φ is a faithful irreducible representation of the group A over the field k .

(ii) It follows from theorem 127.3 of [3] that \hat{k}^* is a direct product of quasicyclic q -groups, where q runs through the set of all prime numbers except p . Then it is not difficult to note that there is a group homomorphism $\varphi : A \rightarrow \hat{k}^*$ such that $\text{Ker}\varphi = H$. Therefore, φ is

an irreducible representation of the group A over the field k such that $\text{Ker}\varphi = H$. \square

Theorem 5.1. *Let G be a soluble group of finite rank and let k be a field. If the group G has a faithful irreducible representation over the field k then $\text{Soc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $\text{Soc}(G)$ by conjugations, and $\text{char}k \notin \pi(\text{Soc}(G))$.*

Proof. Since the group G has finite rank, all minimal abelian normal subgroups of G are finite and the assertion follows from Proposition 4.3. \square

Theorem 5.2. *Let G be a soluble group of finite rank and let k be a not locally finite field. The group G has a faithful irreducible representation over the field k if and only if $\text{Soc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $\text{Soc}(G)$ by conjugations, and $\text{char}k \notin \pi(\text{Soc}(G))$.*

Proof. If the group G has a faithful irreducible representation over the field k then, by theorem 5.1, $\text{Soc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $\text{Soc}(G)$ by conjugations, and $\text{char}k \notin \pi(\text{Soc}(G))$.

Suppose now that $\text{Soc}(G)$ meets the conditions of theorem. Then it follows from proposition 4.3 that $\text{Soc}(G)$ has a subgroup H such that $\text{Soc}(G)/H$ is a locally cyclic p' -group, where $\text{char}k = p$, and H contains no nontrivial G -invariant subgroups. By lemma 5.1(i), $j\text{Soc}(G)$ has a faithful irreducible representation over k and, by lemma 5.1(ii), $\text{Soc}(G)$ has an irreducible representation φ over k such that $\text{Ker}\varphi = H$. By Proposition 3.2, for any nontrivial normal subgroup N of G either $j\text{Soc}(G) \cap N \neq 1$ or $\text{Soc}(G) \cap N \neq 1$. Then it follows from proposition 4.2 that G has a faithful irreducible representation over k . \square

We also obtained a criterion of existence of faithful irreducible representations of soluble groups of finite rank over a locally finite field under some additional conditions.

Theorem 5.3. *Let G be a soluble group of finite rank and let k be a locally finite field. Suppose that $\text{Sp}(B) \not\subseteq \{\text{char}k\}$ for any nontrivial abelian torsion-free normal subgroup B of G . Then the group G has a faithful irreducible representation over the field k if and only if $\text{Soc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module, where the group G acts on $\text{Soc}(G)$ by conjugations, and $\text{char}k \notin \pi(\text{Soc}(G))$.*

Proof. If the group G has a faithful irreducible representation over the field k then, by theorem 5.1, $\text{Soc}(G)$ is a locally cyclic $\mathbb{Z}G$ -module,

where the group G acts on $Soc(G)$ by conjugations, and $chark \notin \pi(Soc(G))$.

Suppose now that $Soc(G)$ meets the conditions of the theorem. Then it follows from proposition 4.3 that $Soc(G)$ has a subgroup H_1 such that $Soc(G)/H_1$ is a locally cyclic p' -group, where $chark = p$, and H_1 contains no nontrivial G -invariant subgroups. By proposition 3.1, $jiSoc(G)$ has a subgroup H_2 such that $jiSoc(G)/H_2$ is a locally cyclic p' -group, where $chark = p$, and H_2 contains no nontrivial G -invariant subgroups. By lemma 5.1, $Soc(G)$ has an irreducible representation φ_1 over k such that $Ker\varphi_1 = H_1$ and $jiSoc(G)$ has an irreducible representation φ_2 over k such that $Ker\varphi_2 = H_2$. By Proposition 3.2, for any nontrivial normal subgroup N of G either $jiSoc(G) \cap N \neq 1$ or $Soc(G) \cap N \neq 1$. Then it follows from proposition 4.2 that G has a faithful irreducible representation over k . \square

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